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New representation for the odderon wave function

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Abstract

New representation of the odderon wave function is derived, which is convergent in the whole impact parameter plane, and provides the analytic form of the quantization condition for the integral of motion q_3. A new quantum number, triality, was identified independently in each sector. This, together with the choice of the conformal basis allows for simple calculation of eigenvalues of a wide class of operators.

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1 Introduction

The resummed scattering amplitudes in perturbative QCD grow with the energy in a qualitative agreement with the Regge theory expectations. Till recently most of the activity has been concentrated around the leading Regge trajectory with the vacuum quantum numbers – the BFKL pomeron [1]. Perturbative QCD predicts also an existence of the object with the same quantum numbers except of the C parity – the odderon [2, 3, 4], which controls the energy behavior of the difference of the cross-sections, $\sigma_{pp}^{\rm tot}-\sigma_{p\bar{p}}^{\rm tot}\sim s^{\alpha_O-1}$, with α_O being the odderon intercept. Contrary to the BFKL pomeron, the intercept of the odderon trajectory has been calculated only recently [5, 6] which generated renewed interest in the subject [7, 8, 9]. In this letter we present an alternative derivation of the odderon wave function. Present approach leads to simpler (although equivalent) final expression. In addition, it offers a possibility of a more direct calculation of the odderon intercept.

In the perturbative QCD approach, the odderon appears as a compound state of three gluons bound together by a nontrivial QCD interaction [2]. The odderon wave function satisfies the Schrödinger equation with the effective QCD Hamiltonian $\mathbb{H}_3$ describing quantum mechanical 3-body system with nearest neighbor interaction on a two-dimensional plane of transverse gluon coordinates $\vec{\rho}=(x,y)$. The odderon intercept, α_O-1, is given by the maximal eigenvalue of the Hamiltonian and the corresponding eigenfunction determines the odderon wave function $\chi_O(\vec{\rho}_1,\vec{\rho}_2,\vec{\rho}_3)$.

The effective QCD Hamiltonian, $\mathbb{H}_3$, possesses a number of remarkable properties which allow to solve the odderon problem exactly [3, 10]. Firstly, it exhibits the holomorphic separability

$$\mathbb{H}_3=\mathcal{H}_3+\overline{\mathcal{H}}_3\,,\qquad [\mathcal{H}_3,\overline{\mathcal{H}}_3]=0, \tag{1}$$

with the operators $\mathcal{H}_3=H_{12}+H_{23}+H_{31}$ and $\bar{\mathcal{H}}_3$ acting on the holomorphic and antiholomorphic gluon coordinates, $\rho_k=x_k+iy_k$, and $\bar{\rho}_k=x_k-iy_k$ $(k=1,2,3)$, respectively. The two-body holomorphic Hamiltonian is given by

$$H_{ik}=\psi(J_{ik})+\psi(1-J_{ik})-2\psi(1)\,,\qquad J_{ik}(1-J_{ik})\equiv L_{ik}^2=-\rho_{ik}^2\partial_i\partial_k. \tag{2}$$

Secondly, in each sector two additional operators exist, which commute with the Hamiltonians $\mathcal{H}_3$ and $\bar{\mathcal{H}}_3$

$$q_2\equiv L^2=L_{12}^2+L_{23}^2+L_{31}^2\,,\qquad q_3=i\rho_{12}\rho_{23}\rho_{31}\partial_1\partial_2\partial_3, \tag{3}$$

with similar expressions for the antiholomorphic operators. The conserved charge, q_2, is given by the quadratic Casimir operator of the $SL(2)$ group and is related to the invariance of $\mathcal{H}_3$ under projective transformations $\rho\to(a\rho+b)/(c\rho+d)$ with $ad-bc=1$. The existence of the additional conserved charge q_3 leads to a complete integrability of the QCD Hamiltonian $\mathbb{H}_3$, and plays a crucial role in our consideration.

Thanks to the projective invariance of the Hamiltonian $\mathbb{H}_3$, the wave function $\chi(\rho_k,\bar{\rho}_k)$ can be classified according to the irreducible representation of the $SL(2,\mathbb{C})$ group. The

choice of a particular representation is dictated by additional physical requirement that one has to impose on the properties of the odderon state. As in the case of the BFKL pomeron, one chooses the appropriate representation as the unitary principal series of the $SL(2,\mathbb{C})$. It is straightforward to show that under this choice the eigenvalues of the Hamiltonian $\mathbb{H}_3$ (including the value of the odderon intercept) take *finite* real values [10]. The corresponding eigenfunctions are parameterized by conformal weights $h = \frac{1}{2}(1+m) - i\nu$ and $\bar{h} = 1 - h^*$ and have the following general form

$$\chi(\rho_k, \bar{\rho}_k) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^h \left(\frac{\bar{\rho}_{12}}{\bar{\rho}_{10}\bar{\rho}_{20}}\right)^{\bar{h}} \Psi(x, \bar{x}). \tag{4}$$

Here, $h - \bar{h} = m$ and $h + \bar{h} = 1 - 2i\nu$ are integer Lorentz spin and the scaling dimension of the state, respectively, $\vec{\rho}_0$ is the center-of-mass coordinate of the state and $\Psi(x, \bar{x})$ is some function of the anharmonic ratio $x = \frac{\rho_{12}\rho_{30}}{\rho_{10}\rho_{32}}$ with $\rho_{ij} = \rho_i - \rho_j$ and similarly for $\bar{x}$. The Bose symmetry of the odderon state implies that $\chi(\rho_k, \bar{\rho}_k)$ has to be a completely symmetric function of gluon coordinates.

For a given h and $\bar{h}$ and any $\Psi(x, \bar{x})$, the function (4) diagonalizes the $SL(2,\mathbb{C})$ Casimir operators q_2 and $\bar{q}_2$ and the corresponding eigenvalues are given by $q_2 = h(1-h)$ and $\overline{q}_2 = \bar{h}(1-\bar{h})$. The explicit form of $\Psi(x, \bar{x})$ should be found from the condition for χ to be an eigenstate of the QCD Hamiltonian (1). Due to a complete integrability, the eigenproblem for $\mathbb{H}_3$ becomes equivalent to a simpler condition for the function (4) to be a simultaneous eigenfunction of the conserved charges q_3 and $\bar{q}_3$. Since these operators act independently on the holomorphic and antiholomorphic coordinates, $\Psi(x, \bar{x})$ has the following general form

$$\Psi(x, \bar{x}) = \sum_{\lambda, \overline{\lambda}} f_{\lambda\overline{\lambda}}\, \Phi_\lambda(x)\, \overline{\Phi}_{\overline{\lambda}}(\bar{x}), \tag{5}$$

where sum goes over the eigenfunctions Φ_λ ($\bar{\Phi}_{\bar{\lambda}}$) of the conserved charge q_3 ($\bar{q}_3$) in the holomorphic (antiholomorphic) sectors and $f_{\lambda\overline{\lambda}}$ are some numerical coefficients. Although the dynamics in x and $\bar{x}-$coordinates is independent of each other, the two sectors are tied together through the condition that the wave function $\chi(\rho_k, \bar{\rho}_k)$ is a single-valued function on the two-dimensional $\rho-$plane. It is this condition that allows to establish the quantization of the conserved charges q_3 and $\bar{q}_3$ and uniquely fix the expansion coefficients $f_{\lambda\overline{\lambda}}$ entering into (5). In this way, the exact spectrum of q_3 and $\bar{q}_3$ was recently derived in [6] and found to agree with the earlier asymptotic WKB expressions [11].

In Sect. 3 we derive a new analytical quantization condition of q_3 that agrees with the both approaches. Our approach is based on the observation that in addition to q_2 and q_3 there exists the new quantum number in each sector. Namely the Hamiltonian (1) is invariant under the cyclic permutations of the reggeon coordinates independently in the holomorphic and antiholomorphic sectors. Denoting the generators of the corresponding transformations as P and $\bar{P}$, we shall choose $\Phi_\lambda(x)$ in (5) to be simultaneous eigenfunctions of q_2 and q_3 possessing a definite "triality" λ ($\lambda^3 = 1$) with respect to

cyclic permutations P. This, together with the conformal basis introduced in the next section, allows for simple calculation of eigenvalues of a wide class of operators including the QCD Hamiltonian.

2 Conformal basis and cyclic symmetry

The requirement for the wave function (4) to diagonalize the conserved charge q_3 leads to the 3rd order ordinary differential equation of the Fuchs type for $\Psi(x,\bar{x})$ as a function of x. It proves convenient to analyze this equation by expanding $\Psi(x,\bar{x})$ over the conformal basis spanned by the functions $\phi_\alpha(x)\phi_{\bar{\alpha}}(\bar{x})$ which, in the main channel (123), have the same conformal weights h and $\bar{h}$ as the wave function Ψ and, in addition, have a definite conformal weights α and $\bar{\alpha}$ in the (12)−subchannel [2]

$$\hat{L}^2\phi_\alpha(x) = h(h-1)\phi_\alpha(x)\,, \qquad \hat{L}^2_{12}\phi_\alpha(x) = \alpha(\alpha-1)\phi_\alpha(x). \tag{6}$$

It is straightforward to show that the solution of (6) is given by a hypergeometric series [3]

$$\phi_\alpha(x) = x^{\alpha-h}F(\alpha,\alpha-h;2\alpha;x), \tag{7}$$

and one finds similar expressions in the antiholomorphic sector. As mentioned in the Introduction, this choice of basis, together with the cyclic symmetry discussed below, allows a rather simple representation for the large class of operators.

Using the relation $q_3 = -i[L^2_{23}, L^2_{31}]$ one verifies that the conserved charge $\hat{q}_3 = U^{-1}q_3U$ is given in the conformal basis by, an infinite-dimensional, three-diagonal matrix

$$i\hat{q}_3\phi_\alpha(x) = A(\alpha)\phi_{\alpha-1}(x) + B(\alpha)\phi_{\alpha+1}(x), \tag{8}$$

with

$$A(\alpha) = \alpha(\alpha-1)(h-\alpha)\,, \quad B(\alpha) = -A(\alpha)\frac{(h+\alpha)(h+\alpha-1)}{4(2\alpha+1)(2\alpha-1)}.$$

The eigenfunction of the operator $\hat{q}_3$ is given by a linear combination of the basis functions that one can effectively represent in the form of the contour integral (we omit the indices for simplicity)

$$\Phi(x) = \int_\Gamma \frac{d\alpha}{2\pi i}\, C(\alpha)\phi_\alpha(x)\,. \tag{9}$$

Here $C(\alpha)$ is some coefficient function having (an infinite) number of poles in the complex $\alpha-$plane and the integration goes over some contour Γ encircling all singularities of $C(\alpha)$. If $C(\alpha)$ has a pole at $\alpha = \alpha_0$, then its contribution to $\Phi(x)$ scales as x^{α_0} at small x. Consequently, requiring $\Phi(x)$ to be an eigenfunction of $\hat{q}_3$ and applying (8), one finds that $C(\alpha)$ should have two additional poles situated at $\alpha = \alpha_0 \pm 1$ for any

[2] Here we introduced the notation for the "hatted" operators $\hat{L}^2_{jk}$ that act on x and are related to L^2_{jk} through the similarity transformation, $\hat{L}^2_{jk} = U^{-1}L^2_{jk}U$ with $U = (\rho_{12}/\rho_{10}\rho_{20})^h$.

[3] Similar conformal basis was previously introduced in [9, 12].

α_0 except of those satisfying $A(\alpha_0) = 0$, or equivalently $\alpha_0 = 0\,,1\,,h$. Each solution to this equation generates a string of poles shifted by 1 along the real axis. The fact, that the first two strings become degenerate starting from $\alpha = 1$ gives rise to a double pole, or equivalently, generates logarithmic singularity at $x \sim 0$. An additional degeneracy occurs when conformal weight of the state h takes integer positive values. In this case, the third series becomes degenerate with the first two and gives rise to triple poles. In what follows we will not consider this case separately. Summarizing, we find that for arbitrary values of h (except of integer values) $C(\alpha)$ admits the following expansion

$$C(\alpha) = \sum_{k=0}^{\infty} \frac{c_k}{\alpha - k} + \frac{a_k}{(\alpha-k)^2} + \frac{b_k}{\alpha - k - h}, \quad a_0 = 0, \tag{10}$$

and the contour Γ in (9) is such that all poles (and only poles) saturate the integral.

Substituting (10) into (9), and using (8), one finds that for $\Phi(x)$ to be an eigenfunction of $\hat{q}_3$ the residue coefficients (a_k, b_k, c_k) have to satisfy the following three-term recurrence relations

$$iq_3 a_k = a_{k+1} A(k+1) - a_{k-1} B(k-1)\,, \tag{11}$$

$$iq_3 b_k = b_{k+1} A(k+h+1) - b_{k-1} B(k+h-1)\,, \tag{12}$$

$$iq_3 c_k = \left(c_{k+1} + a_{k+1}\frac{d}{dk}\right) A(k+1) - \left(c_{k-1} + a_{k-1}\frac{d}{dk}\right) B(k-1), \tag{13}$$

$$a_0 = b_{-1} = c_{-1} = 0.$$

This system determines all coefficients a_k, b_k and c_k in terms of the three initial ones, which were chosen as b_0, c_0 and c_1. A freedom in choosing the initial conditions corresponds to the existence of the three linearly independent solutions to the third order differential equation for the eigenfunctions $\Phi_\lambda(x)$ of the operator $\hat{q}_3$ in the holomorphic sector. The general expression for (9) looks like

$$\Phi_\lambda(x) = c_0(\lambda)\phi^a(x) + b_0(\lambda)\phi^b(x) + c_1(\lambda)\phi^c(x), \tag{14}$$

where λ enumerates three different eigenfunctions and the functions ϕ^a, ϕ^b and ϕ^c are defined as three independent solutions to the recurrence relations Eqs.(11)-(13) corresponding to three different choices of the initial conditions, $(c_0 = 1, b_0 = 0, c_1 = 0)$, $(c_0 = 0, b_0 = 1, c_1 = 0)$ and $(c_0 = 0, b_0 = 0, c_1 = 1)$, respectively

$$\begin{aligned}
\phi^a(x) &= \sum_{k=0}^{\infty} c_k \phi_k(x) + a_k \partial_k \phi_k(x), \quad c_0 = 1,\ c_1 = 0, \\
\phi^b(x) &= \sum_{k=0}^{\infty} b_k \phi_{k+h}(x), \quad b_0 = 1\,, \\
\phi^c(x) &= \sum_{k=0}^{\infty} c_k \phi_k(x), \quad c_0 = 0,\ c_1 = 1\,.
\end{aligned} \tag{15}$$

To fix uniquely the coefficients $c_0(\lambda)$, $b_0(\lambda)$ and $c_1(\lambda)$ we explore an additional *discrete* symmetry of the holomorphic Hamiltonian $\mathcal{H}_3$ and the charges q_2 and q_3 with respect to the cyclic permutations of the holomorphic gluon coordinates

$$P\,\chi(\rho_1,\rho_2,\rho_3)=\chi(\rho_2,\rho_3,\rho_1)\,,\qquad P^3=1\,,\qquad [\mathcal{H}_3,P]=[q_3,P]=0\,. \tag{16}$$

To take the full advantage of the symmetry properties, we choose the three independent functions $\Phi_\lambda(x)$ to be eigenfunctions of the "triality" operator P and identify λ as a corresponding eigenvalue

$$\hat{P}\,\Phi_\lambda(x)=\lambda\Phi_\lambda(x)\,,\qquad \lambda=1,\,\mathrm{e}^{\,2\pi i/3},\,\mathrm{e}^{\,-2\pi i/3}\,. \tag{17}$$

Under the cyclic permutations of holomorphic coordinates their anharmonic ratio x transforms as $x\to 1/(1-x)\to 1-1/x$ and the above condition, together with $\hat{P}^2\Phi_\lambda=\lambda^2\Phi_\lambda$, translates into

$$\begin{aligned}\lambda\Phi_\lambda(x)&=e^{-2\pi ih/3}(-1/x)^h\Phi_\lambda(1/(1-x)),\\ \lambda^2\Phi_\lambda(x)&=e^{-4\pi ih/3}(-1/x)^h(x-1)^h\Phi_\lambda(1-1/x).\end{aligned} \tag{18}$$

Replacing $\Phi_\lambda(x)$ by its expression (14), it is readily seen that these equations can be used to eliminate two of the initial parameters , say b_0/c_0 and c_1/c_0 for each λ. With this choice the remaining freedom reduces to a multiplicative normalization c_0. We choose $c_0=1$ for each value of λ. The relations (18) should hold for any x and a particularly simple choice of x will be discussed later.

A note on the choice of unique branches in (18) is in order. Even though the complete wave function (5) must be single-valued (see later), the eigenfunctions $\Phi_\lambda(x)$ are in general multi-valued and the consistent choice of cuts and phases is required. Eqs.(18) correspond to the choice $|\arg(x)|<\pi,\ \arg(-1/x)=\pi-\arg(x),\ |\arg(x-1)|<\pi$.

Identical considerations can be performed to construct the basis functions $\bar{\Phi}_{\bar\lambda}(\bar x)$ in the antiholomorphic sector with $\bar\lambda$ being an eigenvalue of the operator of cyclic permutations of the antiholomorphic coordinates $\bar{P}\chi(\bar\rho_1,\bar\rho_2,\bar\rho_3)=\chi(\bar\rho_2,\bar\rho_3,\bar\rho_1)$. One should notice that although the operators P and $\bar P$ act independently on the holomorphic and antiholomorphic gluon coordinates, respectively, $[P,\bar P]=0$, it is the operator $P\bar P$ that generates the cyclic permutations of the gluons in the two-dimensional $\vec\rho=(\rho,\bar\rho)-$plane. In addition, the relation $\bar\rho=\rho^*$ implies that the definition of the cuts and phases of the basis functions $\bar{\Phi}_{\bar\lambda}(\bar x)$ coincide with those of $(\Phi_\lambda(x))^*$.

3 Construction of the wave function

Having defined the functions $\Phi_\lambda(x)$ and $\bar\Phi_{\bar\lambda}(\bar x)$ we look for the wave function $\Psi(x,\bar x)$ in the form (5) with λ and $\bar\lambda$ being cubic roots of unity and $f_{\lambda\bar\lambda}$ being a 3×3 matrix. Thus constructed $\Psi(x,\bar x)$ is a simultaneous eigenfunction of the integrals of motion

$\hat{q}_k$ and $\bar{\hat{q}}_k$ for arbitrary $f_{\lambda\bar{\lambda}}$. Let us now impose the condition that the odderon wave function should be Bose symmetric single-valued function of gluon coordinates on the 2-dim impact parameter $\vec{\rho}-$plane.

The Bose symmetry can be implemented in two steps. First, requiring the symmetry under the cyclic permutations, $P\bar{P}\Psi(x,\bar{x}) = \Psi(x,\bar{x})$, and using (18) we find the relation between holomorphic and antiholomorphic trialities

$$\lambda\bar{\lambda} = \exp\left(-2\pi i(h-\bar{h})/3\right) = \exp(-2\pi i m/3), \tag{19}$$

with m being the Lorentz spin of the state. Hence, for cyclic symmetric wave function $\Psi(x,\bar{x})$ the matrix $f_{\lambda\bar{\lambda}}$ has only three nonvanishing entries $f_\lambda \equiv f_{\lambda,\lambda^2 \mathrm{e}^{-2\pi i m/3}}$

$$\Psi(x,\bar{x}) = \sum_{\substack{\lambda=1,\mathrm{e}^{\pm 2\pi i/3} \\ \bar{\lambda}=\lambda^2\mathrm{e}^{-2\pi i m/3}}} f_\lambda\, \Phi_\lambda(x)\bar{\Phi}_{\bar{\lambda}}(\bar{x}). \tag{20}$$

Finally, the full Bose symmetry is achieved through the symmetrization $(\rho_1,\rho_2,\rho_3) \to (\rho_2,\rho_1,\rho_3)$, or equivalently $x \to x/(x-1)$

$$\Psi_{\mathrm{sym}}(x,\bar{x}) = \Psi(x,\bar{x}) + \Psi\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right). \tag{21}$$

The expression (20) still contains three unknown parameters f_λ. To fix them we impose the condition for $\Psi(x,\bar{x})$ to be a single-valued function around three singular points $x=\bar{x}=0$, $x=\bar{x}=1$ and $x=\bar{x}=\infty$ corresponding to the limit when any two of the reggeons are approaching each other. Due to the cyclic symmetry of $\Psi(x,\bar{x})$ it is sufficient to consider only one of the singular points, say $x=\bar{x}=0$. To this end, we substitute (14) into (20) and examine the asymptotic behaviour of the functions (15) around $x=0$

$$\begin{aligned}
\phi^a(x) &= x^{-h}\left(1+\frac{iq_3}{(h-1)}x\log x+...\right), \\
\phi^b(x) &= x^0(1+...), \\
\phi^c(x) &= x^{-h+1}(1+...),
\end{aligned} \tag{22}$$

where ... denote terms suppressed by a power of x. Similar expressions hold in the antiholomorphic sector [4]. It is easy to see, using (22), that for $q_3 \neq 0$ and arbitrary complex (but not integer) h and $\bar{h}$ the only bilinear combination of the functions $\phi^{(a,b,c)}(x)$ and $\bar{\phi}^{(a,b,c)}(\bar{x})$, that is well-defined around $x=\bar{x}=0$, has the following general form [5]

$$\begin{aligned}
\Psi(x,\bar{x}) = {} & f_{cc}\phi^c(x)\bar{\phi}^c(\bar{x}) + f_{bb}\phi^b(x)\bar{\phi}^b(\bar{x}) \\
& + f_{ac}\phi^a(x)\bar{\phi}^c(\bar{x}) + f_{ca}\phi^c(x)\bar{\phi}^a(\bar{x}),
\end{aligned} \tag{23}$$

[4]Incidentally Eq.(22) establishes one to one correspondence with the solutions of the third order differential equation considered in [6].

[5]For $q_3=0$ the Bose symmetric wave function of the three gluon compound state is given by the sum of the 2-gluon BFKL wave functions, $\chi(\vec{\rho}_1,\vec{\rho}_2,\vec{\rho}_3) = \chi_0(\vec{\rho}_1,\vec{\rho}_2)+\chi_0(\vec{\rho}_2,\vec{\rho}_3)+\chi_0(\vec{\rho}_3,\vec{\rho}_1)$ with $\chi_0(\vec{\rho}_1,\vec{\rho}_2) = \left(\rho_{12}/\rho_{10}\rho_{20}\right)^h\left(\bar{\rho}_{12}/\bar{\rho}_{10}\bar{\rho}_{20}\right)^{\bar{h}}$. However, this state is degenerate and has a zero $SL(2)-$norm.

provided that $f_{ac}/f_{ca} = q_3(\bar{h}-1)/(\bar{q}_3(h-1))$. The latter condition ensures that $\sim \ln(x)$ and $\sim \ln(\bar{x})$ terms enter with the same coefficients. At the same time, replacing $\Phi(x)$ and $\bar{\Phi}(\bar{x})$ in (20) by their expressions (14) one obtains 9 different bilinear combinations of the functions $\phi^{(a,b,c)}(x)$ and $\bar{\phi}^{(a,b,c)}(\bar{x})$. Matching them into (23) one finds

$$f_{cc} = \sum_\lambda f_\lambda c_1(\lambda)\bar{c}_1(\bar{\lambda}), \quad f_{bb} = \sum_\lambda f_\lambda b_0(\lambda)\bar{b}_0(\bar{\lambda}),$$

$$f_{ac} = \sum_\lambda f_\lambda \bar{c}_1(\bar{\lambda}), \quad f_{ca} = \sum_\lambda f_\lambda c_1(\lambda),$$

provided that λ and $\bar{\lambda}$ are related through (19) and

$$\frac{q_3}{h-1}\sum_\lambda f_\lambda \bar{c}_1(\bar{\lambda}) = \frac{\bar{q}_3}{\bar{h}-1}\sum_\lambda f_\lambda c_1(\lambda)\,. \tag{24}$$

We recall that throughout the paper we are using the normalization $c_0(\lambda) = \bar{c}_0(\bar{\lambda}) = 1$. The requirement that the remaining 5 combinations do not appear in (23) leads to the following conditions

$$q_3\bar{q}_3\sum_\lambda f_\lambda = \sum_\lambda f_\lambda \bar{b}_0(\bar{\lambda}) = \sum_\lambda f_\lambda b_0(\lambda) = \sum_\lambda f_\lambda \bar{b}_0(\bar{\lambda})c_1(\lambda) = \sum_\lambda f_\lambda b_0(\lambda)\bar{c}_1(\bar{\lambda}) = 0 \tag{25}$$

The parameters $b_0(\lambda)$ and $c_1(\lambda)$ together with their antiholomorphic counterparts depend on the charges q_3 and $\bar{q}_3 = q_3^*$ and, as we show in the next section, can be uniquely defined from the triality conditions (18). The system of the equations (24) and (25) on the coefficients f_λ is overcompleted and its consistency conditions provide the quantization of q_3.

4 Quantization conditions

The odderon corresponds to the eigenstate $\Psi(x,\bar{x})$ with the maximal energy and is expected to have a zero Lorentz spin, $m=0$, and pure imaginary value of the charge q_3. This allows us to restrict further analysis to the lowest representation

$$h = \bar{h} = \frac{1}{2} + i\nu\,, \qquad \mathrm{Re}\, q_3 = 0\,. \tag{26}$$

Let us now determine the coefficients $b_0(\lambda)$ and $c_1(\lambda)$ from the triality conditions Eqs.(18). To this end we examine the asymptotic behavior of the both sides of Eqs.(18) in the limit $x \to 0$ and use the known asymptotics of the conformal basis functions ϕ_α around $x = 0, 1$ and ∞

$$\phi_\alpha(x) \stackrel{x\to 0}{\sim} x^{\alpha-h}\,, \quad \stackrel{x\to 1}{\sim} \frac{\Gamma(2\alpha)\Gamma(h)}{\Gamma(\alpha+h)\Gamma(\alpha)}\,, \quad \stackrel{x\to\infty+i\epsilon}{\sim} \mathrm{e}^{\,i\pi(\alpha-h)}\frac{\Gamma(2\alpha)\Gamma(h)}{\Gamma(\alpha+h)\Gamma(\alpha)}\,. \tag{27}$$

Then, Eqs.(18) are transformed into

$$\begin{aligned}\lambda \mathrm{e}^{-i\pi h/3} &= \alpha(iq_3) + c_1(\lambda)\beta(iq_3) + b_0(\lambda)\gamma(iq_3), \\ \lambda^2 \mathrm{e}^{i\pi h/3} &= \alpha(-iq_3) + \left(\frac{\pi}{h-1}q_3 - c_1(\lambda)\right)\beta(-iq_3) + b_0(\lambda)\gamma(-iq_3)\mathrm{e}^{i\pi h},\end{aligned} \tag{28}$$

where the functions α, β and γ depend only on the charge q_3 and are defined through the series

$$\begin{aligned}\alpha(iq_3) &= \sum_{k=0}^{\infty}\left(c_k + a_k\frac{d}{dk}\right)\frac{\Gamma(h)\Gamma(2k)}{\Gamma(k)\Gamma(k+h)}\Big|_{c_0=1,\ c_1=0}, \\ \beta(iq_3) &= \sum_{k=1}^{\infty} c_k\frac{\Gamma(h)\Gamma(2k)}{\Gamma(k)\Gamma(k+h)}\Big|_{c_0=0,\ c_1=1}, \\ \gamma(iq_3) &= \sum_{k=0}^{\infty} b_k\frac{\Gamma(h)\Gamma(2(k+h))}{\Gamma(k+h)\Gamma(k+2h)}\Big|_{b_0=1},\end{aligned} \tag{29}$$

with the coefficients (a_k, b_k, c_k) satisfying the recurrence relations Eqs. (11)–(13). The antiholomorphic coefficients $\bar{b}_0$ and $\bar{c}_1$ obey similar equations and are related to their holomorphic counterpart as

$$\bar{c}_0 = 1\,, \qquad \bar{b}_0(\bar{\lambda}) = b_0(\lambda)\mathrm{e}^{i\pi h}\,, \qquad \bar{c}_1(\bar{\lambda}) = -c_1(\lambda) + \frac{\pi q_3}{h-1}. \tag{30}$$

Using these relations one can resolve the system of equations (24) and (25) to obtain the explicit expressions for the coefficients f_λ as

$$f_{\lambda_i} = \varepsilon_{ijk}\frac{c_1(\lambda_j) - c_1(\lambda_k)}{b_0(\lambda_i)}, \tag{31}$$

with $i, j, k = 1, 2, 3$ and $\lambda_k = \exp(2i\pi k/3)$. Then, the quantization condition of q_3 takes a simple form (for $q_3 \neq 0$)

$$f_{\lambda_1} + f_{\lambda_2} + f_{\lambda_3} = 0\,. \tag{32}$$

We recall that the coefficients $c_1(\lambda)$ and $b_0(\lambda)$ entering (31) and (30) are solutions of the system of linear equations (28). Solving (28) it becomes straightforward to express f_λ in terms of the functions $\alpha(iq_3)$, $\beta(iq_3)$ and $\gamma(iq_3)$ defined in (29). In this way one evaluates the sum (32) and identifies its zeros as quantized values of the odderon charge q_3. The solutions to the quantization conditions are shown in Fig. 1.

We find that in an agreement with the WKB analysis [11], for any real ν defining the conformal weight of the state, (26), the quantized q_3 takes a (infinite) series of discrete values $q_3 = q_3(\nu, k)$ that can be enumerated by a positive integer k with $k = 1$ corresponding to the smallest $|q_3| \neq 0$. Moreover, considering the dependence of q_3 on real ν and integer positive k we observe that $q_3(\nu, k)$ is a smooth function of ν for given k and, therefore, the quantized values of q_3 form a family of curves parameterized by an integer k. The lowest three curves corresponding to $k = 1\,, 2\,,$ and 3 are shown in Fig. 1.

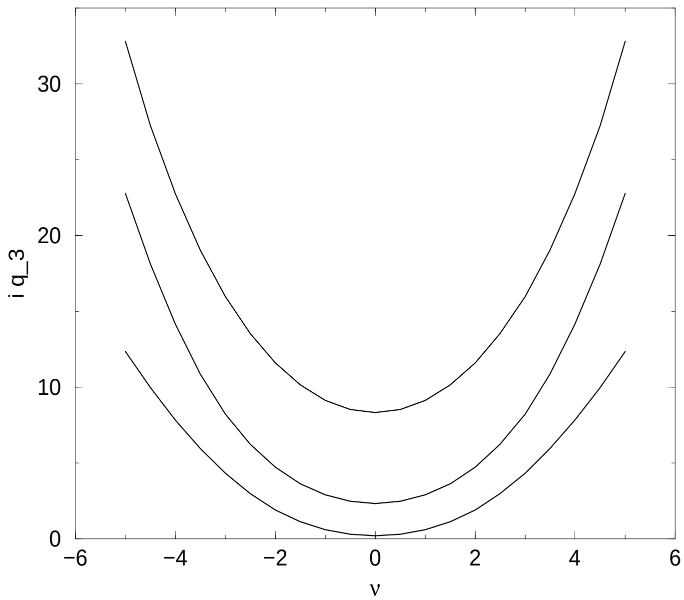

Figure 1: The flow of the quantized values of the charge $q_3 = q_3(\nu, k)$ with the conformal weight $h = 1/2 + i\nu$ for the lowest three trajectories $k = 1, 2, 3$.

5 Special case: $h = 1/2$

1. *The eigenvalues.* Additional important simplification occurs for the smallest absolute value of the conformal weight, $h = \bar{h} = \frac{1}{2}$, or equivalently $\nu = 0$. Using the fact that α, β and γ are real functions of iq_3 one finds that in this case the quantization condition (32) can be rewritten in one of the following *equivalent* forms

$$\begin{aligned} \gamma(iq_3) &= 0\,, \\ 2i\pi q_3 \beta^2(iq_3) &= \beta(-iq_3), \\ \alpha(iq_3)\beta(-iq_3) &= -\alpha(-iq_3)\beta(iq_3). \end{aligned} \tag{33}$$

Considering the first relation and using the definition (29) one obtains the quantization condition for q_3 at $h = \bar{h} = \frac{1}{2}$ in the form

$$\gamma(iq_3) = \sum_{k=0}^{\infty} 4^k b_k(q_3) = 0\,. \tag{34}$$

That is, the eigenvalues of q_3 are the zeroes of the generating function of the b_k coefficients considered as a function of q_3. Introducing new coefficients $\hat{b}_k = (-4)^k k(k^2 - 1/4) b_k$, such that $4iq_3 b_k = (-4)^{-k} \left(\hat{b}_{k+1} - \hat{b}_{k-1}\right)$, one gets

$$\gamma(iq_3) = \hat{b}_{2\infty+1}(q_3) - \hat{b}_{2\infty}(q_3) = 0\,. \tag{35}$$

No.	iq_3	$iq_3^{\rm as}$
1	0.20525750608820	0.2052575
2	2.34392106326404	2.343918
3	8.32634590161324	8.307685

Table 1: Quantization of q_3.

Here $\hat{b}_{2\infty+1}$ and $\hat{b}_{2\infty}$ are limiting values of the coefficients $\hat{b}_k$ for odd and even k, respectively. To find their values one considers the three term recurrence relations for $\hat{b}_k$

$$4iq_3\hat{b}_k = k\left(k^2 - \frac{1}{4}\right)\left[\hat{b}_{k+1} - \hat{b}_{k-1}\right], \qquad \hat{b}_0 = 0\,, \quad \hat{b}_1 = 1\,. \tag{36}$$

Their solution can be found using a 2×2 transfer matrix as

$$\begin{pmatrix} \hat{b}_{n+1} \\ \hat{b}_n \end{pmatrix} = \prod_{k=1}^{n}\begin{pmatrix} v_k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_k = \frac{4iq_3}{k(k^2-1/4)}. \tag{37}$$

Then, it is straightforward to show that $\hat{b}_{2\infty+1}$ and $\hat{b}_{2\infty}$ are the smooth, real functions of iq_3 and solutions to (35) define an infinite set of discrete real positive values of iq_3. At small iq_3 one applies the recurrence relations (36) to obtain the first few terms of the expansion of $\gamma(q)$

$$\begin{aligned}\gamma(q) &= 1 - 8\ln 2\, q + 3.4066793\, q^2 - 0.6271540\, q^3 + 0.0494534\, q^4, \\ &- 0.0020214\, q^5 + 0.0000483\, q^6 - 0.0000007\, q^7 + \mathcal{O}(q^8)\end{aligned} \tag{38}$$

where the expansion coefficients starting from $\mathcal{O}(q^2)-$term can be expressed in terms of the hypergeometric series.

The expansion (38) can be used to obtain the lowest values of quantized q_3. In Table 1 we quote the first three eigenvalues. High precision solutions of Eq.(32) are displayed in the first column [6]. These numbers fully agree with the results obtained in [6] and subsequent more precise estimates [7]. The first three zeroes of the small q_3 expansion, Eq.(38), are quoted in the second column. They agree extremely well with the precise results for the lowest state. Even for the third state the small q_3 approximation works qualitatively.

2. The wave function. For $h = \bar{h} = 1/2$ additional simplification occurs for the odderon wave function as well. Firstly, the basis of $SL(2,\mathbb{C})$ harmonics, (7), reduces to a simple power

$$\phi_\alpha(x) = (4u)^{\alpha-1/2}\,, \qquad u = \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}, \tag{39}$$

[6]In order to reach quoted precision it is numerically advantageous to solve Eqs.(18) at the Lagrange point $x_L = \exp(i\pi/3)$ point (c.f. Section 6) where the convergence of $k-$ summation is exponential.

which suggests an important role of the new variable u and its antiholomorphic counterpart $\bar{u}$ defined in a similar way. In new variables the completely symmetric odderon wave function reads, $\varphi = \arg(u)$

$$\Psi_{\rm sym}(x,\bar{x}) = \sum_{m\geq 0, m={\rm even}}^{\infty} 2\cos(m\varphi) S_m(|u|), \tag{40}$$

$$S_m(|u|) = \sum_{r\geq |m|, r={\rm even}}^{\infty} C_{mr}|u|^{r-1} + B_{mr}|u|^r + 2A_{mr}|u|^{r-1}\log(4|u|),$$

where the coefficients A , B and C are given by

$$\begin{aligned} A_{mr} &= 4^{r-1} f_{ac}\bar{c}_k^{(c)} = 4^{r-1} f_{ca} c_k^{(c)}, B_{mr} = 4^r f_{bb} b_k \bar{b}_n, \\ C_{mr} &= 4^{r-1}(f_{cc} c_k^{(c)} \bar{c}_n^{(c)} + f_{ac} c_k^{(a)} \bar{c}_n^{(c)} + f_{ca} c_k^{(c)} \bar{c}_n^{(a)}), \end{aligned} \tag{41}$$

$k = (r+m)/2$, $n = (r-m)/2$ and $c_k^{(a,c)}$ denote the expansion coefficients entering the expansion of the functions $\phi^a(x)$ and $\phi^c(x)$, respectively, defined in Eq.(15).

This representation is convergent in the unit circle $|u| \leq 1$ onto which the cut complex plane of x is mapped under (39) with both edges of the cut $1 < x < \infty$ transformed onto the circumference $|u| = 1$. It is worth noting that the function $S_m(\rho)$ satisfies the relations $S_{odd}(1) = 0$ and $dS_{even}(1)/d\rho = 0$, which guarantee smoothness of $\Psi_{\rm sym}$ across the cut $1 < x < \infty$.

Since the basic conditions imposed on the compound eigenfunction (20) (uniqueness and Bose symmetry), as well as the resulting structure (23), are the same as in Ref.[6] it is natural that the resulting wave functions are identical with those constructed there. We have found a complete numerical agreement for the three lowest states. However the present approach is advantageous in the two respects. First, it gives much simpler (although equivalent) quantization of q_3, and second, the representation (15) is *convergent* in the whole transverse plane while earlier expression required the analytical continuation between the three domains of x.

6 Observables and sum rules

Having constructed the eigenvalues and the eigenfunctions we turn now to the observables. A large class of operators, relevant to the odderon problem, consists of the $SL(2,\mathbb{C})$ invariant operators symmetric under the cyclic permutations and having a pairwise structure, say $\mathcal{O} = f(L_{12}^2) + f(L_{23}^2) + f(L_{31}^2)$. Their eigenvalues can be readily extracted using the expansion (9). Even though the three terms entering $\mathcal{O}$ do not commute with each other, the symmetry of the wave function (17) under P allows us to calculate the action of $\mathcal{O}$ on the states $\Phi_\lambda(x)$. Using the identities $f(L_{23}^2) = Pf(L_{12}^2)P^2$ and $f(L_{31}^2) = P^2 f(L_{12}^2)P$ and replacing $\Phi_\lambda(x)$ by its expansion (9) we get

$$\mathcal{O}\Phi_\lambda(x) = (1 + \lambda^2 P + \lambda P^2) f(L_{12}^2)\Phi_\lambda(x) = \int_\Gamma \frac{d\alpha}{2\pi i}\, C(\alpha) f(\alpha(\alpha-1))\varphi_\alpha(x;\lambda), \tag{42}$$

where

$$\varphi_\alpha(x;\lambda) = \phi_\alpha(x)+\lambda^2 e^{-2\pi i h/3}(-1/x)^h\phi_\alpha\left(\frac{1}{1-x}\right)+\lambda e^{-4\pi i h/3}(-1/x)^h(x-1)^h\phi_\alpha\left(1-\frac{1}{x}\right). \tag{43}$$

The integration in (42) reduces to the discrete sum over poles of the coefficient functions at $\alpha = k,\ k+h$. By an appropriate symmetric choice of x this formula can be yet simplified. At the Lagrange point [7], $x_L = \exp(i\pi/3)$, such that $x_L = 1/(1-x_L) = 1-1/x_L$ one finds that

$$\varphi_\alpha(x_L;\lambda) = \phi_\alpha(x_L)(1+\lambda+\lambda^2) = 3\phi_\alpha(x_L), \tag{44}$$

for $\lambda = 1$ and $\varphi_\alpha(x_L;\lambda) = 0$ for $\lambda = \exp(\pm 2i\pi/3)$. Therefore,

$$\mathcal{O}\,\Phi_\lambda(x_L) = 3\delta_{\lambda,1} f(L_{12}^2)\,\Phi_\lambda(x_L) = 3\delta_{\lambda,1}\int_\Gamma \frac{d\alpha}{2\pi i}\,C(\alpha)f(\alpha(\alpha-1))\phi_\alpha(x_L)\,. \tag{45}$$

Another advantage of the Lagrange point is that for $h = \bar{h} = 1/2$ the phase of the harmonics $\phi_\alpha(x_L)$ given by Eq.(39) is very simple and the absolute value of $\phi_\alpha(x_L)$ is small, which makes the sum over residues in (45) exponentially convergent.

If $\Phi_\lambda(x)$ is the eigenfunction of $\mathcal{O}$, then the relation (45) allows for calculating the corresponding eigenvalue. On the other hand, comparing the x-dependence of the both sides of the general formula (42) one could verify whether $\Phi_\lambda(x)$ is the *eigenfunction* of $\mathcal{O}$. In particular this is the case for $\mathcal{O} = L^2 = L_{12}^2 + L_{23}^2 + L_{31}^2$. Moreover, since by the construction $\Phi_\lambda(x)$ diagonalizes the Casimir operator L^2, and the resulting eigenvalue is known to be $h(h-1)$, the above equations provide useful sum rules to test our construction of the wave function. We verified that for $h = \bar{h} = 1/2$ these sum rules are satisfied to high accuracy independently of x.

Analogous sum rules hold for the complete eigenfunction (23). However, while the sum rules (42) for $\Phi_\lambda(x)$ are satisfied for arbitrary q_3, similar relations for the eigenfunction $\Psi(x,\bar{x})$ [8]

$$(L^2 + \bar{L}^2)\Psi(x,\bar{x})/\Psi(x,\bar{x}) = 2\,\mathrm{Re}\,h(h-1)\,, \tag{46}$$

are satisfied only for the quantized values of q_3, since in arriving at Eqs.(23) and (40) we have used the quantization conditions (24) and (25). In fact, a high numerical precision quoted in the first column of Table 1 was required to satisfactorily reproduce rhs of (46) at $h = \bar{h} = 1/2$ for the first three states in a large range of x.

It becomes straightforward to apply the above method to calculate the eigenvalues of the Hamiltonian, (1). Using the cyclic symmetry, its action on the wave function $\Phi_\lambda(x)$ can be reduced at the Lagrange point to that of $H_{12} + \bar{H}_{12}$ only. Moreover, because of (2), Eq.(42) with $f(\alpha) = \psi(\alpha) + \psi(1-\alpha) - 2\psi(1)$ would apply. Similarly one could calculate the action of the Hamiltonian $\mathbb{H}_3$ on the wave function (20). We note, however, that the two body holomorphic energy, $f(\alpha)$, has additional poles at

[7] The Lagrange point corresponds to three bodies (gluons) forming an equilateral triangle - a configuration well known in astronomy.

[8] With the analog of the rhs of Eq.(42) given essentially by Eq.(40) with obvious modifications.

integer values of the conformal weight α, which could interfere with the singularities of the coefficient function in (45). Although these poles cancel out in the sum of the holomorphic and antiholomorphic energies $f(\alpha)+f(\bar{\alpha})$, provided that $\bar{\alpha}$ and α are taking values in the principal series representation of the $SL(2,\mathbb{C})$, one has to introduce an additional prescription to separate them from the singularities of the coefficient function $C(\alpha)$.

7 Conclusions

New representation of the odderon wave function was derived, which is convergent in the whole impact parameter plane, and provides the analytic form of the quantization condition for the integral of motion q_3. Solving these quantization conditions we have found that the quantized values of q_3 form smooth trajectories parametrized by an integer. Our results are in agreement with the findings of Ref.[6] and earlier WKB expressions [11]. A new quantum number (triality) was identified independently in each sector. This, together with the choice of the conformal basis, allows for simple calculation of the eigenvalues of a wide class of operators. Angular momentum sum rules were derived and shown to provide a useful test of the whole scheme. The odderon Hamiltonian also belongs to the above class, however its action on the conformal basis requires additional studies.

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